Periodic measures and Wasserstein distance for analysing periodicity of time series datasets

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I. Introduction

Many important results in ergodic theory of stochastic dynamical systems have been obtained for

invariant measures and stationary processes.

We have established ergodic theory (any invariant set has (invariant) measure 0 or 1) for

(1) Random periodic processes (Feng-Z. (JDE 2020))

(2) Random quasi-periodic processes (Feng-Qu-Z. (JDE 2021), 2023+a)

(3) Processes under nonlinear expectations (Feng-Z. (SIMA 2021), Feng-Wu-Z. (SPA 2021), Feng-Huang-Liu-Z. 2023+)

Observations and questions:

• Time series is seen as a pathwise process.

• Periodicity: DFT approach and its shortfalls.

• Could the random periodic processes and their ergodic theory be developed to deal with time series datasets? Is it better?

This is answered now in part by results in Feng-Liu-Z. CNSNS 2023.

II. Random periodic paths and periodic measures

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Polish space $\mathbb{X}$, let $\Phi(t, s) : \Omega \times \mathbb{X} \to \mathbb{X}$ be a stochastic semiflow over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}; (\theta_t)_{t \in \mathbb{R}})$.

Definition 1

(Z.-Zheng (JDE 2009), Feng-Z.-Zhou (JDE 2011), Feng-Z. (JFA 2012)) A random periodic path of period T is an $\mathcal{F}$-measurable map $Y : \mathbb{R} \times \Omega \to \mathbb{X}$ such that for almost all $\omega \in \Omega$,

$$\Phi(t, s, \omega) Y(s, \omega) = Y(t, \omega), \quad \forall s \in R, t \geq s \tag{1}$$

and for any $s \in \mathbb{R}$,

$$Y(s + T, \omega) = Y(s, \theta_T \omega). \tag{2}$$

Define the transition probability of Markovian dynamical system Φ:

$$P(t, s, x, B) := \mathbb{P}(\omega : \Phi(t, s, \omega)x \in B), \qquad \text{for any } B \in \mathcal{B}(\mathbb{X}).$$

Definition 1

(Feng-Z. (JDE 2020), Feng-Wu-Z. (JFA 2016), Feng-Liu-Z. (ZAMP 2017)) A measure valued function $\{\rho_s\}_{s\in R}$ in $\mathcal{P}(\mathbb{X})$ is an <u>entrance measure</u> on $(\mathbb{X}, \mathcal{B})$ if

$$\rho_t(B) = \int_{\mathbb{X}} P(t, s, x, B)\rho_s(dx), \quad s \in \mathbb{R}, \; t \geq s, \; B \in \mathcal{B}(\mathbb{X}). \tag{3}$$

<u>Periodic measure</u>: $\rho_{T+s} = \rho_s$,
<u>Invariant measure</u>: $\rho_s = \rho_0$ for all s.

Quasi-periodic measure: Feng-Qu-Z. JDE (2021)

Theorem 2

(Feng and Z. (JDE 2020))
Random periodic paths " <=> " periodic measures.

III. Ergodicity: homogeneous case

$$\rho_{T+s} = \rho_s, \quad P_t^*\rho_s = \rho_{t+s}, \ t \in R^+. \tag{4}$$

Observation:

$$\bar{\rho} = \frac{1}{T}\int_{[0,T)} \rho_s ds \ \ (\text{or} \sum \ \text{if discrete})$$

is an invariant measure of $\{P_t\}_{t\in\mathbb{R}^+}$. (thus have ergodic theory: any invariant set has measure $\bar{\rho}$ 0 or 1).

Inhomogeneous case

Lifting: define

$$\hat{\mathbb{X}} = [0, T) \times \mathbb{X},$$

and for any $A \in \mathcal{B}(\hat{\mathbb{X}})$, A_r-the r section of A,

$$\hat{P}(t, (s, x), A) = \int_0^T \delta_{t+s \mod T}(dv) P(t + s, s, x, A_v),$$

$$\hat{\rho}_s(A) = \int_0^T \delta_{s \mod T}(dv) \rho_s(A_v).$$

Then one can prove that

$$\hat{P}_t^* \hat{\rho}_s = \hat{\rho}_{t+s}, \quad \hat{\rho}_{s+T} = \hat{\rho}_s.$$

Invariant measure:

$$\bar{\hat{\rho}} = \frac{1}{T} \int_0^T \hat{\rho}_s ds.$$

	weakly mixing	ergodic	ergodic new-1 Feng-Z. JDE 20
Markovian semigroup P_t eigenvalues	1 simple unique on unit circle (**von Neumann**)	1 simple	$\{e^{i\frac{2m\pi t}{T}}\}_{m\in\mathbb{Z}}$ simple only on unit circle
infinitesimal generator $\mathcal{L}$ eigenvalues	0 simple unique on imaginary axis	0 simple	$\{i\frac{2m\pi}{T}\}_{m\in\mathbb{Z}}$ simple only on imaginary axis
convergence to invariant measure	relatively measure 1 set	average (**Birkhoff** ET)	average (PS-mixing)
processes	stationary (aperiodic)	stationary	random periodic

Example 3

There is a unique entrance measure. to

$$dX_t = (X_t - X_t^3 + f(t))dt + dW_t. \tag{5}$$

Periodic Measure: $f(t) = A\cos(\alpha t)$ *(Benzi-Parisi-Sutera-Vulpiani's climate change model (1983)): there is a unique periodic measure and it is ergodic (Feng-Z.-Zhong (JDE 2023)).*

The **uniqueness** *is significant in explaining the* **transitions** *between the two wells (two climates) as otherwise there should be two periodic measures instead of one, together with PDE for the expected switching time (Feng-Z.-Zhong (Physica D 2021)), provided a rigorous proof for the result proposed by Parisi et al.*

Quasi-periodic measure: $f(t) = A_1\cos(\alpha_1 t)\cdots + A_n\cos(\alpha_n t)$ *(quasi-periodic measure, extended BPSV climate change model, Feng-Qu-Z. (2023+a)), unique and ergodic*

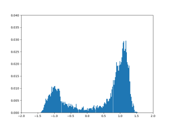
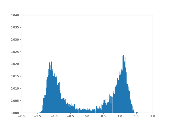
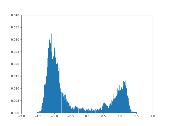


Figure: $\hat{\rho}_0$



$\hat{\rho}_{625}$



$\hat{\rho}_{1250}$

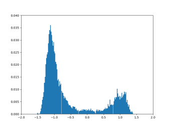
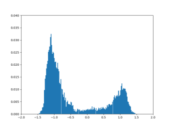
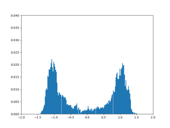


Figure: $\hat{\rho}_{1875}$



$\hat{\rho}_{2500}$



$\hat{\rho}_{3125}$

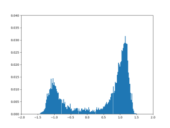
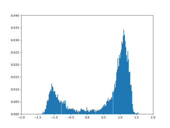


Figure: $\hat{\rho}_{3750}$



$\hat{\rho}_{4375}$

Contributed to the following work:

–*climate dynamics*: Chekroun, Simonnet and Ghil (Physica D (2011))

–*stochastic bifurcations*: Wang (Nonlinear analysis 2014)

–*random attractors*: Bates-Lu-Wang (Physica D 2014)

–*stochastic resonance*: Cherubini-Lamb-Rasmussen-Sato (Nonlinearity 2017), Feng-Zhao-Zhong (JDE 2023, Physica D 2021), Feng-Liu-Zhao (JCAM 2021, 2023+), Feng-Qu-Z. (2023+a)

–*stochastic horseshoe*: Huang-Lian-Lu (2019/2021)

–*random almost periodic solutions*: Cheng-Liu (2019), Zhang-Zheng (2019), Raynaud de Fitte (SD 2020)

Impacts continued

–*stochastic oscillation*: Engel-Kuehn (CMP 2021)

–*Linear response of SDEs and homogenizations* : Branicki-Uda (RMS 2021), Uda (SPA 2021)

–*SFDEs, SDEs, SPDEs, McKean-Vlasov*: Gao-Yan (2018), Song-Song-Zhang (SIMA 2020), Dong-Zhang-Zheng (2021, 2022), Liu-Lu (2021, 2022), Wu-Yuan (JTP 2023), Bao-Wu (2022), Feng-Qu-Z. (2023+b)...

–*Large deviations*: Gao-Liu-Sun-Zheng (2022)

–*Synchronizations*: M Engel, G Olicón-Méndez, N Unger, S Winkelmann (2022)

–*Alternations: Engel-Kuehn (CMP 2021), Sun-Zheng (CMB 2023)*

IV. Non-randomness of the period

Many people will have expected that the period of a random periodic path might be random rather than deterministic. Note in the Definition 1, as a basic assumption of this paper, the period T is a deterministic number rather than a random variable. Note

"random periodic"

does not necessarily mean

"random period".

Observation (Feng-Zhao (JDE 2020)): Setting $\phi(s,\omega) := Y(s,\theta_{-s}\omega)$, then $Y(s+T,\omega) = Y(s,\theta_T\omega)$ for all $s \in \mathbb{R}$ if and only if $\phi(s+T,\omega) = \phi(s,\omega)$ and note also that for almost all $\omega \in \Omega$,

$$\Phi(t,\omega)\phi(s,\omega) = \phi(s+t,\theta_t\omega), \text{ for any } t,s \in \mathbb{R}, \tag{6}$$

is equivalent to (1).
Consider a random path Y of Φ. It is a function $\mathbb{R} \times \Omega \to \mathbb{X}$ satisfying $\Phi(t,\theta_s\omega)Y(s,\omega) = Y(s+t,\omega)$ for any $t \in \mathbb{R}^+$, $s \in \mathbb{R}$. Consider $\phi(s,\omega) := Y(s,\theta_{-s}\omega)$. Assume

$$T(\omega) := \inf\{t > 0 | \phi(s+t,\omega) = \phi(s,\omega) \text{ for all s } \} \tag{7}$$

exists.

Theorem 2

Assume a measurable function $\phi : \mathbb{R} \times \Omega \to \mathbb{X}$ exists such that (6) holds for a.e. $\omega \in \Omega$, θ is ergodic and $T : \Omega \to \mathbb{R}^+$ defined by (7) exist. If T is positive $\mathbb{P}$-a.s., then it is constant $\mathbb{P}$-a.s.

V. LNN with test period

Let Y be a random periodic path of random dynamical system Φ. If $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_{kT})_{k\in\mathbb{N}})$ is ergodic, then for any $\Gamma \in \mathcal{B}(\mathbb{X})$, $t \in \mathbb{R}$,

$$\frac{1}{K}\sum_{k=0}^{K-1} I_\Gamma\left(Y(t+kT,\omega)\right) \to \mathbb{E}I_\Gamma(Y_t(\cdot)) = \rho_t(\Gamma) \tag{8}$$

as $K \to \infty$ $\mathbb{P}$-a.s. and in $L^2(\Omega, d\mathbb{P})$.
However, the result itself may not be that useful in applications as
• the period T is often unknown
• slight difference of the value T that appears on the left hand side of (8) can result in some significant difference to the convergence of (8).
Thus it is crucial to study (8) for $Y(t+k\tau,\omega)$, where τ could be different from T.

Set $\tilde{\Omega} = [0,T) \times \Omega$, where $T > 0$ is constant and taken as the period of the random periodic path.
Note first for any fixed $t \geq 0$, there exists $m_t \in \mathbb{N}$ and $j_t \in [0,T)$ such that $t = m_t T + j_t$. For any $t \geq 0$, $(s,\omega) \in \tilde{\Omega}$, set

$$\tilde{\Theta}_t(s,\omega) = (j_{t+s}, \theta_{m_{t+s}T}\omega),$$

and for any $A \in \mathcal{B}([0,T)) \otimes \mathcal{F}$, define

$$\tilde{\mathbb{P}}(A) = \frac{1}{T} \int_{[0,T)} \mathbb{P}(A_s) ds,$$

where $A_s := \{\omega \in \Omega : (s,\omega) \in A\}$ being the s-section.

Lemma 3

The map $t \mapsto \tilde{\Theta}_t$ is a semigroup and preserves $\tilde{\mathbb{P}}$.

Theorem 4

There exists a random measure function ρ such that

$$\frac{1}{K}\sum_{k=0}^{K-1}\delta_{Y(s+k\tau,\omega)}(\Gamma)\to\rho_{s,\omega}(\Gamma), \tag{9}$$

$\tilde{\mathbb{P}}$*-a.s. and in* $L^2(\tilde{\Omega}, d\tilde{\mathbb{P}})$*. Moreover,* $\rho_{s,\omega}=\rho_{\tilde{\omega}}=\rho_{\tilde{\Theta}_\tau\tilde{\omega}}$.

Key in the proof:

$$\begin{aligned}\frac{1}{K}\sum_{k=0}^{K-1}\delta_{Y(s+k\tau,\omega)}(\Gamma) &= \frac{1}{K}\sum_{k=0}^{K-1} I_\Gamma\left(Y(j_{s+k\tau}+m_{s+k\tau}T,\omega)\right)\\ &= \frac{1}{K}\sum_{k=0}^{K-1} I_\Gamma\left(Y(j_{s+k\tau},\theta_{m_{s+k\tau}T}\omega)\right)\\ &= \frac{1}{K}\sum_{k=0}^{K-1} I_\Gamma\left(Y(\tilde{\Theta}_{k\tau}\tilde{\omega})\right).\end{aligned}$$

Now we consider the case that τ and T are rationally dependent. Let integers q^*, p^* be co-prime to each other such that

$$q^*\tau = p^*T. \quad (10)$$

Then for all s,

$$s + q^*\tau = j_s + m_{s+q^*\tau}T = s + p^*T \quad (11)$$

and q^* is the smallest of such integer satisfying (11). Following Theorem 4, it is easy to prove the following result.

Theorem 5

Assume assumptions of Theorem 4 and that τ and T are rationally dependent with q^*, p^* defined by (10). If $\theta_{p^*T} : \Omega \to \Omega$ is ergodic, then

$$\frac{1}{K}\sum_{k=0}^{K-1} \delta_{Y(s+k\tau,\omega)}(\Gamma) \to \rho_s(\Gamma)$$

for $\tilde{\mathbb{P}}$-a.s. $\tilde{\omega} \in \tilde{\Omega}$, and in $L^2(\tilde{\Omega}, d\tilde{\mathbb{P}})$ and ρ_s is independent of ω for almost all s.

Proof.

Note

$$\frac{1}{K}\sum_{k=0}^{K-1} I_\Gamma(Y(s+k\tau,\theta_{p^*T}\omega)) = \frac{1}{K}\sum_{k=0}^{K-1} I_\Gamma(Y(s+k\tau+p^*T,\omega))$$
$$= \frac{1}{K}\sum_{k=0}^{K-1} I_\Gamma(Y(s+(k+q^*)\tau,\omega)) = \frac{1}{K}\sum_{k=q^*}^{K+q^*-1} I_\Gamma(Y(s+k\tau,\omega))$$
$$\to \rho_{s,\omega}(\Gamma) \tag{12}$$

$\tilde{\mathbb{P}}$-a.s. and in $L^2(\tilde{\Omega}, d\tilde{\mathbb{P}})$. Here we used Theorem 4 in the above convergence. But $\frac{1}{K}\sum_{k=0}^{K-1} I_\Gamma(Y(s+k\tau,\theta_{p^*T}\omega)) \to \rho_{s,\theta_{p^*T}\omega}$ a.s. by Theorem 4 again. Thus $\rho_{s,\omega} = \rho_{s,\theta_{p^*T}\omega}$ $\tilde{\mathbb{P}}$-a.s. It follows that $\rho_{s,\omega} = \rho_{s,\theta_{p^*T}\omega}$ for almost all $\omega \in \Omega$. It then follows from ergodic theory as $\theta_{p^*T} : \Omega \to \Omega$ preserves $\mathbb{P}$ and is ergodic that $\rho_{s,\omega}$ is independent of ω. □

In many situations, the underlying noise is Brownian motion. In this case the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Wiener space and the measure preserving dynamical system $\theta : I \times \Omega \to \Omega$ is given by $(\theta_t \omega)(s) = W(t+s) - W(t)$.

In Feng-Qu-Z. (Nonlinearity 2020), it was proved that the metric dynamical system given as the shift of Brownian motions is ergodic. The theorem is stated below.

Theorem 6

The canonical dynamical system of Brownian motion $\Sigma = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{T}})$ $(\mathbb{T} = \mathbb{R}^+$ or $\mathbb{R})$ and their discrete dynamical system $\Sigma^T = (\Omega, \mathcal{F}, \mathbb{P}, (\theta_T^n)_{n \in I})$ $(I = \mathbb{N}$ or $\mathbb{Z})$ are ergodic.

Consider two integers $p, q \geq 1$ satisfying $p \leq q$. All the results are still true when $p > q$ with a slight modification of proofs. Define $S = \{0, 1, 2, \cdots, q-1\}$, and a dynamical system on the finite integer field S, $T : S \to S$ by

$$T(i) = (i+p)|q, \qquad i \in S \tag{13}$$

and the trace of i as

$$S(i) = \{T^n(i)|n \in \mathbb{N}\} = \{j \in S | j = i + k_1 p - k_2 q, k_1 \in \mathbb{N}^+, k_2 \in \mathbb{N}^+ \cup \{0\}\},$$

where $i \in S$. The following two lemmas are equivalent to Bézout's identity.

Lemma 7

(1) The integers p, q *are co-prime to each other if and only if* $S(0) = S$.

(2) The integers p, q *have a greatest common divisor* r *if and only if for* $0 \leq i < r$,

$$S(i) = \{i, i + r, i + 2r, \cdots\} \cap S = \{i, i + r, \cdots, i + (q^* - 1)r\},$$

where $q^* = \frac{q}{r}$.

Theorem 8

Assume the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_{kp^*q})_{k\in\mathbb{N}})$ *is ergodic. Then for any* $i \in \mathbb{N}$, $A \in \mathcal{B}(\mathbb{R})$, $p \in \mathbb{N}^+$,

$$p_K^{i,p}(A) := \frac{1}{K}\sum_{k=0}^{K-1} \delta_{Y_{i+kp}(\omega)}(A) \to \frac{1}{q^*}\sum_{u\in S(i)} \rho_u(A) \quad \text{as } K \to \infty,$$

a.s. and in $L^2(\Omega, d\mathbb{P})$*. In particular, when* p, q *are co-prime,*

$$p_K^{i,p}(A) \to \frac{1}{q}\sum_{u=0}^{q-1} \rho_u(A) \quad \text{as } K \to \infty$$

and when $p = q$,

$$p_K^{i,p}(A) \to \rho_i(A) \quad \text{as } K \to \infty,$$

a.s. and in $L^2(\Omega, d\mathbb{P})$.

VII. Quantifying periodicity by Wasserstein distance

Let $d \geq 1$ and $\mathcal{P}(\mathbb{R}^d)$ be the set of all probability measures on $\mathbb{R}^d$. For $\alpha \geq 1$ and $\rho, \nu \in \mathcal{P}(\mathbb{R}^d)$, consider the αth Wasserstein distance between them as

$$W_\alpha(\rho, \nu) := \inf_{\xi \in \mathcal{H}(\rho,\nu)} \left\{ \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^\alpha \xi(dx, dy) \right)^{\frac{1}{\alpha}} \right\},$$

where $\mathcal{H}(\rho, \nu)$ is the set of all probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals ρ and ν.

It is very natural to use the Wasserstein distance to describe the periodicity of a periodic measure and to detect periodicity in a time series dataset, as $i \mapsto W_\alpha(\rho_1, \rho_i)$ is a real valued periodic function.

In this subsection, we will establish the theoretical result on the convergence of empirical distributions in the Wasserstein distance.

In order to prove the main result of this section (Theorem 10), we recall the following result in Fournier-Guillin (2015).
Consider first on $(-1, 1]^d$, denote $\mathcal{P}_l$ as the natural partition of $(-1, 1]^d$ into 2^{dl} disjoint, equal-distance sets. For example, when $d = 1$, $\mathcal{P}_l = \{(-1 + \frac{k}{2^{l-1}}, -1 + \frac{k+1}{2^{l-1}}]\}_{k=0}^{2^l-1}$. To extend to $\mathbb{R}^d \times \mathbb{R}^d$, we introduce $B_0 := (-1, 1]^d$ and $B_n := (-2^n, 2^n]^d \backslash (-2^{n-1}, 2^{n-1}]^d$ for $n \geq 1$. In Fournier-Guillin (2015), the authors proved the following lemma:

Lemma 9

Let $d \geq 1$ and $\alpha > 0$. For all pairs of probability measures ρ, ν on $\mathbb{R}^d$,

$$W_\alpha^\alpha(\rho, \nu) \leq \kappa_{\alpha,d} C_\alpha \sum_{n\geq 0} 2^{\alpha n} \sum_{l \geq 0} 2^{-\alpha l} \sum_{F \in \mathcal{P}_l} |\rho(2^n F \cap B_n) - \nu(2^n F \cap B_n)|, \tag{14}$$

with the notation $2^n F := \{2^n x : x \in F\}$ and where $\kappa_{\alpha,d} := 2^\alpha d^{\alpha/2}(2^\alpha + 1)/(2^\alpha - 1)$ and $C_\alpha := 1 + 2^{-\alpha}/(1 - 2^{-\alpha})$.

Theorem 10

Assume all the conditions of Theorem 8 hold and there exists $\delta > 0$ such that for all t, $\int_{\mathbb{R}} |x|^{\delta+1}\rho_t(dx) < \infty$. Then as $K \to \infty$,

$$\mathbb{E}[W_1(\rho_K^{i,p}, \rho^{i,p})] \to 0.$$

Corollary 11

Assume all the conditions of Theorem 10 hold. There exists a subsequence $K_m \to \infty$ as $m \to \infty$ such that $W_1(\rho_{K_m}^{i,p}, \rho_{K_m}^{j,p}) \to W_1(\rho^{i,p}, \rho^{j,p})$ as $m \to \infty$ for all $i, j \in \{0, 1, \cdots, r-1\}$ a.s.

Example 4

Consider the following stochastic differential equation (SDE),

$$dX(t) = \left(-\pi X(t) + \sin\left(\frac{\pi t}{2}\right) + 1\right) dt + \left(0.1 + 0.6 \sin\left(\frac{\pi t}{5}\right)\right) dW_t \quad (15)$$

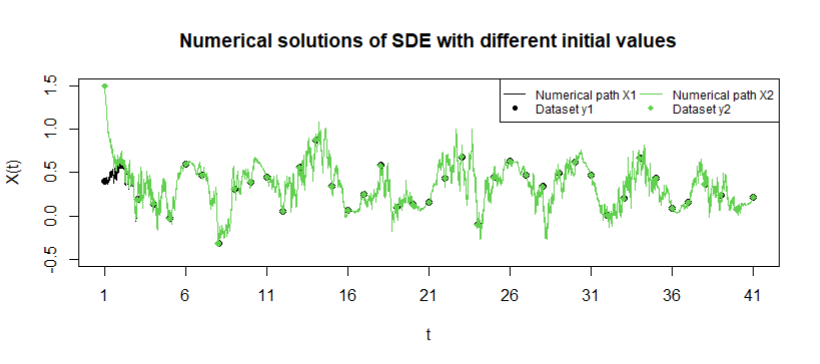
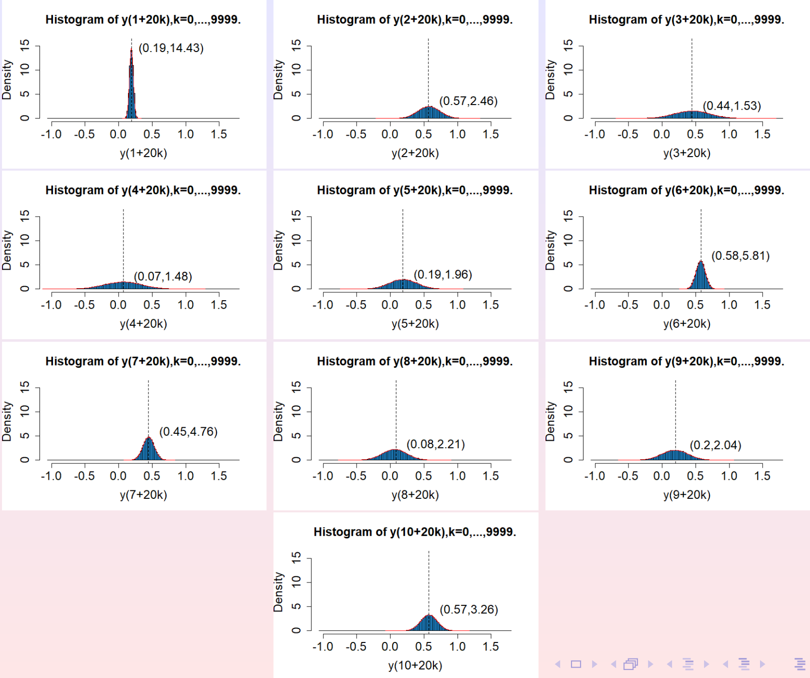
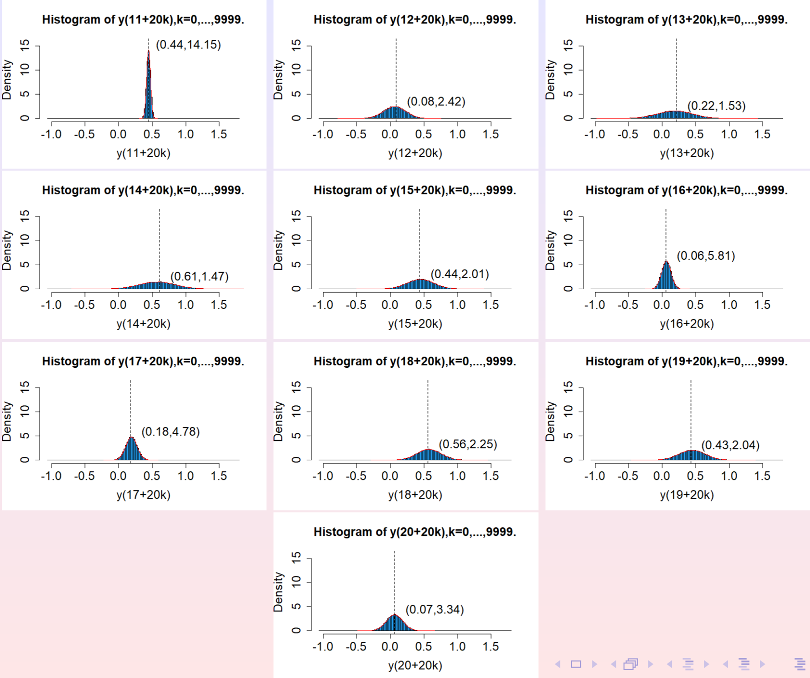


Figure: Numerical simulation of the solution to SDE (15)


Histogram of y(1+20k),k=0,...,9999.
(0.19,14.43)
Density
y(1+20k)
Histogram of y(2+20k),k=0,...,9999.
(0.57,2.46)
Density
y(2+20k)
Histogram of y(3+20k),k=0,...,9999.
(0.44,1.53)
Density
y(3+20k)
Histogram of y(4+20k),k=0,...,9999.
(0.07,1.48)
Density
y(4+20k)
Histogram of y(5+20k),k=0,...,9999.
(0.19,1.96)
Density
y(5+20k)
Histogram of y(6+20k),k=0,...,9999.
(0.58,5.81)
Density
y(6+20k)
Histogram of y(7+20k),k=0,...,9999.
(0.45,4.76)
Density
y(7+20k)
Histogram of y(8+20k),k=0,...,9999.
(0.08,2.21)
Density
y(8+20k)
Histogram of y(9+20k),k=0,...,9999.
(0.2,2.04)
Density
y(9+20k)
Histogram of y(10+20k),k=0,...,9999.
(0.57,3.26)
Density
y(10+20k)
0 5 10 15
-1.0 -0.5 0.0 0.5 1.0 1.5


Histogram of y(11+20k),k=0,...,9999.
(0.44,14.15)
y(11+20k)
Histogram of y(12+20k),k=0,...,9999.
(0.08,2.42)
y(12+20k)
Histogram of y(13+20k),k=0,...,9999.
(0.22,1.53)
y(13+20k)
Histogram of y(14+20k),k=0,...,9999.
(0.61,1.47)
y(14+20k)
Histogram of y(15+20k),k=0,...,9999.
(0.44,2.01)
y(15+20k)
Histogram of y(16+20k),k=0,...,9999.
(0.06,5.81)
y(16+20k)
Histogram of y(17+20k),k=0,...,9999.
(0.18,4.78)
y(17+20k)
Histogram of y(18+20k),k=0,...,9999.
(0.56,2.25)
y(18+20k)
Histogram of y(19+20k),k=0,...,9999.
(0.43,2.04)
y(19+20k)
Histogram of y(20+20k),k=0,...,9999.
(0.07,3.34)
y(20+20k)
Density
0 5 10 15
-1.0 -0.5 0.0 0.5 1.0 1.5

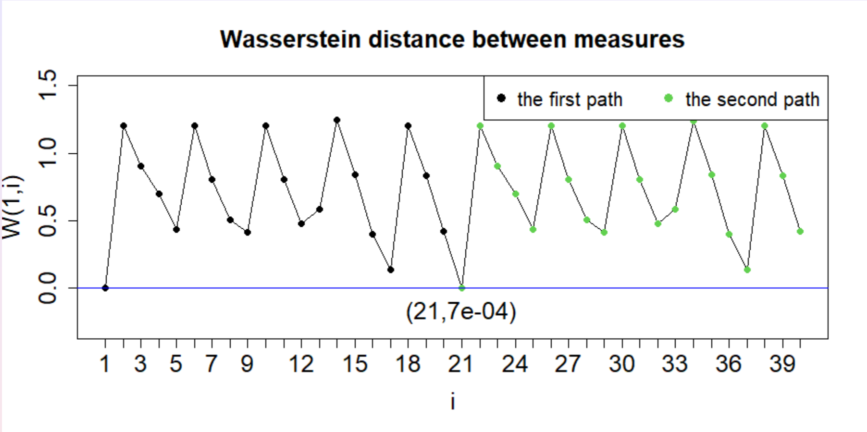


Figure: Wasserstein distance $W(\rho_1, \rho_i)$, $i = 1, 2, \cdots, 40$

VI. The periodic measure and test period LNN algorithm to detect the true period

- Challenging question: the true period q of the random periodic process or time series may be unknown to us.

- The point is that the period, though may be unknown to us, exists in some sets of data.

- Our result gives a way to detect the period using test periods.

Note

(i) $\rho^{i,p}$ depends on the test period p when the true period q is regarded as fixed.

(ii) $i \mapsto \rho^{i,p}$ has periodicity of period r, where r is the greatest common divisor of p and q. If it turns out that

- $i \mapsto \rho^{i,p}$ is aperiodic, then p, q are co-prime.
- If $i \mapsto \rho^{i,p}$ is periodic with period r, then r divides q.
- At $p = q$, $i \mapsto \rho^{i,p}$ has maximum period, in other words, if the period of $i \mapsto \rho^{i,p}$ is maximised at certain p, then $q = p$.

In this context, the question remains to ask is: how do we know the period of $i \mapsto \rho^{i,p}$ is maximised at certain p?

• We assume as a prior knowledge that $q \leq Q$ for certain integer Q. Note that any integer number can be decomposed as

$$q = r_1^{n_1} r_2^{n_2} \cdots r_m^{n_m}, \tag{16}$$

where $r_1 < r_2 \cdots < r_m$ are prime numbers and $n_1, n_2, \cdots, n_m$ are positive integers.

• We start from the test period $p = 2$. For large N, consider the map of empirical measure approximation

$$i \mapsto \rho_K^{i,p} = \frac{1}{K} \sum_{k=0}^{K-1} \delta_{Y_{i+kp}(\cdot)}. \tag{17}$$

• If initially (17) is approximately an invariant measure, it means 2 or any power of 2 is not in the decomposition (16).
• If (17) appears to have period 2, then it means that 2 is a factor of q. Then we can continue to test $p = 2^2, 2^3, \cdots, 2^{r_1}$, for $r_1 \le [\log_2 Q]$, then stop at one step $p = 2^{j_0+1}$ when 2^{j_0+1} is no longer the period of $i \mapsto \rho_K^{i,\cdot}$ (but 2^{j_0} is). In this case we know that 2^{j_0} is a factor of q and j_0 is the maximum power of factor $2^{\cdot}$ of the number q.
• We can decide any of the prime numbers and their powers appearing in the decomposition (16) by applying the above method for other possible p (noting the prior knowledge $q \le Q$ here), thus eventually find the period q.

Example 4 (continued)

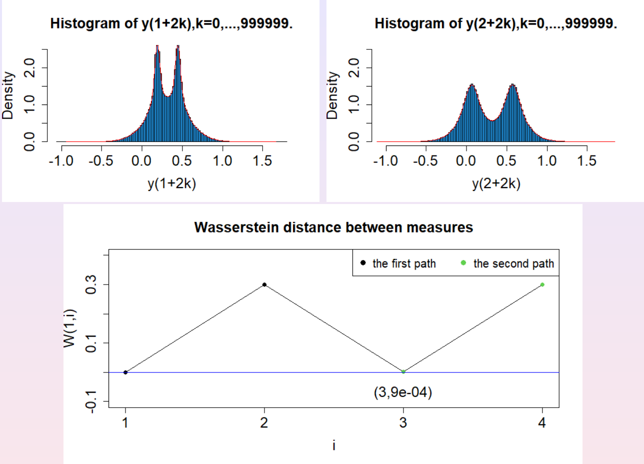
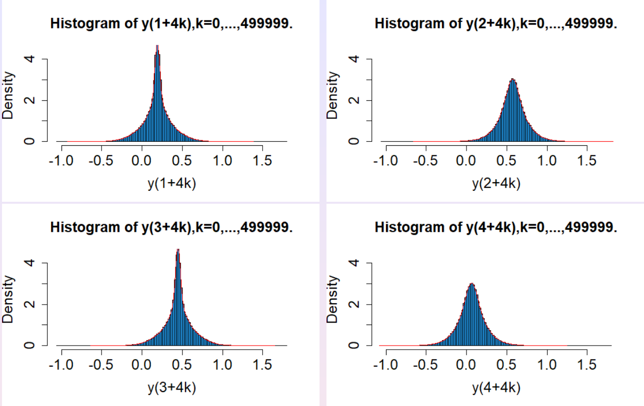
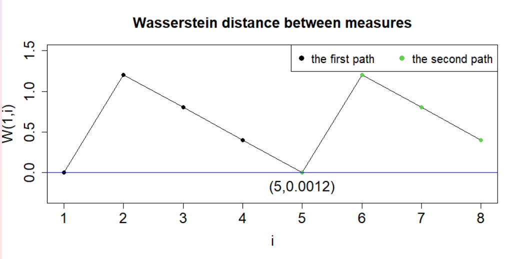


Figure: Analysis of empirical measures of sub-datasets $\{y(i+kp)\}_{k=0}^{K-1}$ when $p=2$


Histogram of y(1+4k),k=0,...,499999.
Density
y(1+4k)
Histogram of y(2+4k),k=0,...,499999.
Density
y(2+4k)
Histogram of y(3+4k),k=0,...,499999.
Density
y(3+4k)
Histogram of y(4+4k),k=0,...,499999.
Density
y(4+4k)


Wasserstein distance between measures
the first path
the second path
W(1,i)
(5,0.0012)
i

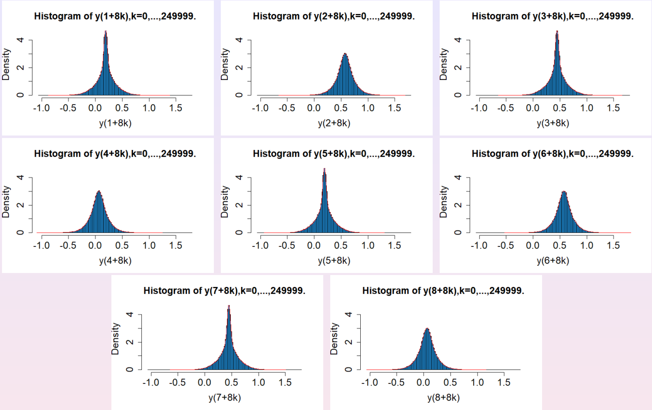


Figure: Analysis of measures of sub-datasets $\{y(i+kp)\}_{k=0}^{K-1}$ when $p = 8$

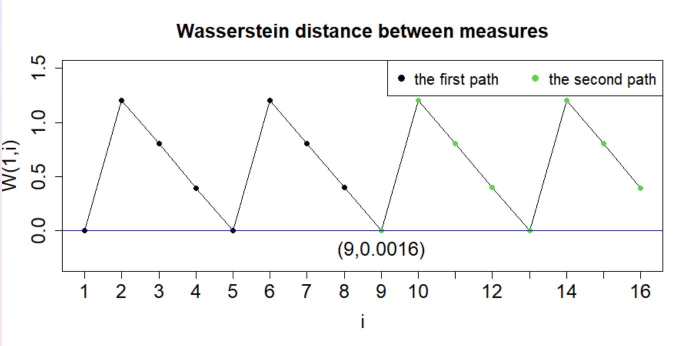
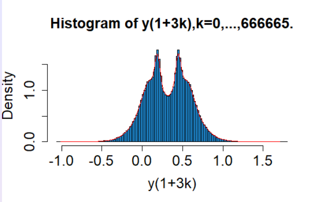
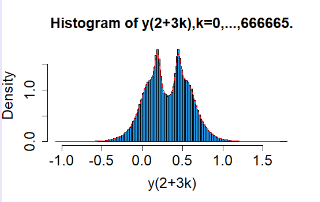
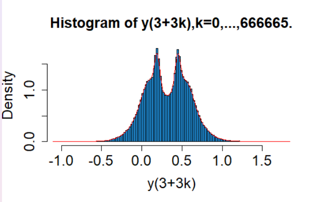
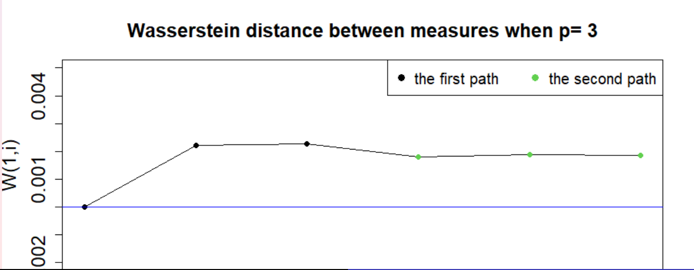


Figure: $i \mapsto W(\rho_K^{1,8}, \rho_K^{i,8}),\ i = 1, 2, \cdots, 16$


Histogram of y(1+3k),k=0,...,666665.
Density
0.0
1.0
-1.0 -0.5 0.0 0.5 1.0 1.5
y(1+3k)


Histogram of y(2+3k),k=0,...,666665.
Density
0.0
1.0
-1.0 -0.5 0.0 0.5 1.0 1.5
y(2+3k)


Histogram of y(3+3k),k=0,...,666665.
Density
0.0
1.0
-1.0 -0.5 0.0 0.5 1.0 1.5
y(3+3k)


Wasserstein distance between measures when p= 3
the first path
the second path
W(1,i)
0.004
0.001
002

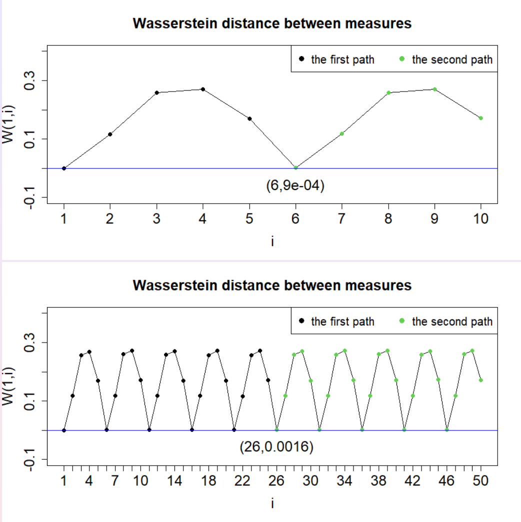


Figure: The Wasserstein distances $i \mapsto W(\rho_K^{1,p}, \rho_K^{i,p})$ for different $p = 5, 25$

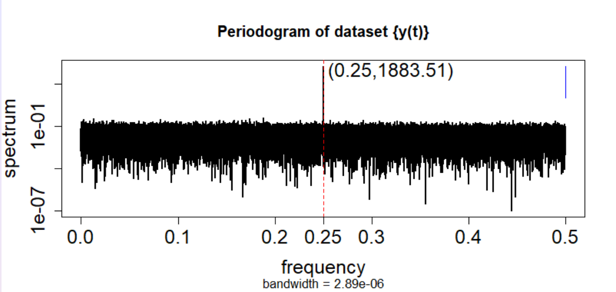


Figure: Periodogram of dataset generated from SDE (15

)

We use the function "*spec.pgram*$(\cdot)$" in the package *STATS* in R Language in which it calculates the periodogram using a fast Fourier transform, the periodicity reflecting to the discrete dataset is 4 on the mean trend and 10 on the noise fluctuations respectively. This suggests that the DFT method only detects the periodicity of the mean trend but not sensitive to the periodicity of the noise!

Example 5

Consider the following SDE,

$$dX(t) = \left(-\pi X(t) + 0.1 \sin\left(\frac{\pi t}{2}\right) + 1\right) dt + \left(0.1 + 10 \sin\left(\frac{\pi t}{5}\right)\right) dW_t, \tag{18}$$

with $X(0) = x$. In this example, the periodicity of mean is weak and the noise perturbation is strong.

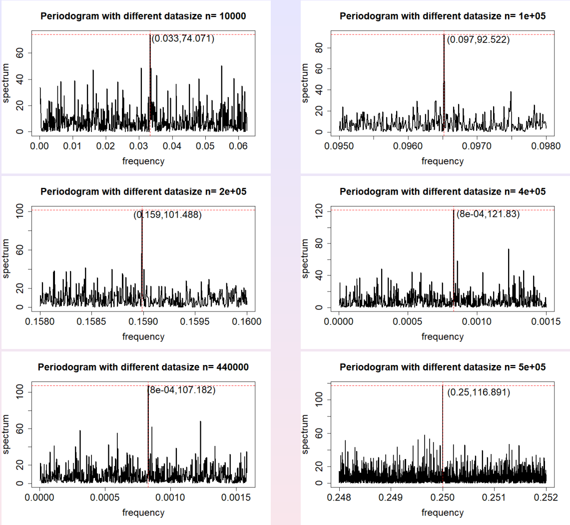


Figure: Periodograms of datasets $\{y(t)\}_{t=1}^{n}$ for different datasize n

Remark: the DFT is not robust.

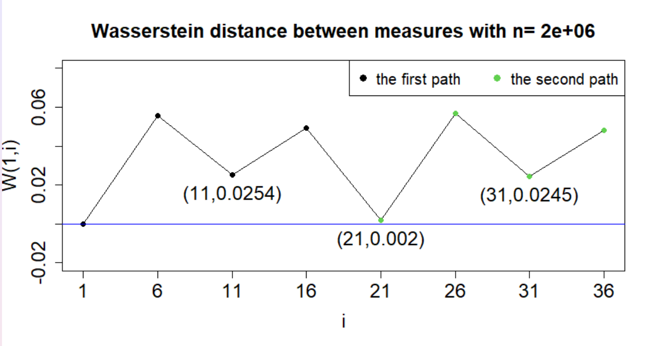


Figure: Figures of the Wasserstein distance $W(\rho_1, \rho_i)$ for $i = 1, \cdots, 40$

Example 6

The dataset $\{y(t)\}$ we used in this example is the monthly average maximal temperature in Oxford. It contains 1954 monthly data for almost 163 years from Jan. 1853 to Oct. 2015, which are computed from the daily maximal temperature records. Part of the data is plotted in Figure 17.

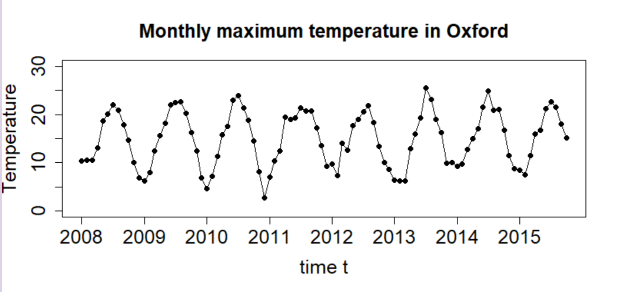


Figure: *Daily maximum temperature in Central England*

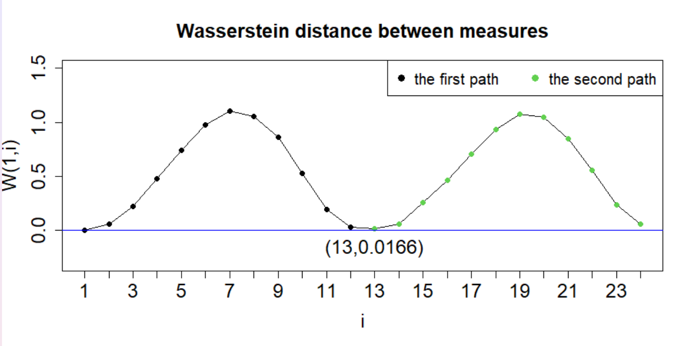


Figure: Wasserstein distance $i \mapsto W(\rho_1, \rho_i)$ for $i = 1, \cdots, 24$

THANK YOU!